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Gaussian Fluctuations for the Magnetization of Lee–Yang Ferromagnets at Zero External Field

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We consider the fluctuations of the block spin magnetization normalized by the square root of the considered number of spins in a block Λ for Lee–Yang ferromagnets. It is established that the fluctuations are Gaussian when $\Lambda \uparrow \mathbb{Z}^d$ at zero external field whenever the susceptibility is finite (i.e., above the critical temperature) and converges to the second derivative of the pressure at zero field. The validity of this fluctuation-dissipation condition is known to hold for a large class of Lee–Yang models, including, for instance, classical Heisenberg ferromagnets.

KEY WORDS: Gaussian fluctuations; Lee–Yang; infinite divisibility.

1. INTRODUCTION

Gaussian fluctuations have been a subject of considerable interest rather recently. Several authors have been able to prove the validity of this result for the magnetization of ferromagnets. For completeness let us briefly review the literature.

One of the first proofs of such a general result was given by Baker and Krinsky.[3] Their method rests on the GKS and Gaussian inequalities and can therefore be applied at zero external field only. A second very interesting step was achieved by Iagolnitzer and Souillard,[4] who proved the Gaussian fluctuations of the magnetization using the Lee–Yang theorem at nonzero external field. Their proof proceeds with a one-component ferromagnet, but can easily be extended to multicomponent ferromagnets. A very powerful method has also been proposed by

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Newman,[5,6] but it relies on the FKG inequalities and is therefore restricted to one-component ferromagnets.

As one of the most powerful methods dealing with multicomponent ferromagnets rests on the Lee–Yang theorem,[13–16,23] it would clearly be desirable to extend Iagolnitzer and Souillard's method in order to be able to study the fluctuations of the magnetization at zero external field for models with several spin components. This is precisely the aim of this paper.

In Sections 2 and 3 we study the fluctuations of the magnetization (suitably renormalized) for Lee–Yang ferromagnets at zero external field. It is established that these fluctuations are Gaussian as the number of spins becomes infinite whenever the susceptibility is finite (i.e., above T_c) and converges to the second derivative of the pressure at zero field. Let us stress that the proof of this fluctuation-dissipation relation only requires some knowledge about the two-point function at zero external field. This property has been in particular established by De Coninck and Dunlop[9] for a large class of Lee–Yang multicomponent ferromagnets using Simon and Gaussian inequalities.[18–22] Our method of proof is new and relies on a detailed study of the integrated two-point function at nonzero external field viewed as a characteristic function (Fourier transform of a positive measure). This property follows from the connection established in Refs. 7 and 8 between the Lee–Yang theorem and the infinitely divisible characteristic functions. Some general remarks on the validity of our result are developed in Section 4.

2. THE PROBLEM

Let us first specify the models in which we are interested. Consider a block $\Lambda \subset \Lambda' \subset \mathbb{Z}^d$. We associate a spin variable $\boldsymbol{\sigma}_i$ (with one, two, or three components) at each vertex of Λ'. The corresponding family of finite-volume Gibbs states are chosen to be

$$d\mu_{\Lambda,\Lambda'}((\boldsymbol{\sigma}_j)_{j\in\Lambda'} \mid \beta, h)$$
$$= Z_{\Lambda,\Lambda'}^{-1}(\beta, h) \exp\left\{\beta \sum_{i,j\in\Lambda'} J_{ij}\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j + h \sum_{k\in\Lambda} \sigma_k^1\right\} \prod_{k\in\Lambda'} d\nu(\boldsymbol{\sigma}_k) \tag{1}$$

where the normalization factor is given by

$$Z_{\Lambda,\Lambda'}(\beta, h) = \int_{\mathbb{R}^{|\Lambda'|}} \exp\left(\beta \sum_{i,j\in\Lambda'} J_{ij}\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j + h \sum_{k\in\Lambda} \sigma_k^1\right) \prod_{l\in\Lambda'} d\nu(\boldsymbol{\sigma}_l) \tag{2}$$

The coefficients $J_{ij} \geqslant 0$ describe the interactions between pairs of spins, β is the inverse temperature, and h is the applied external field on Λ in units of temperature. The interactions J_{ij} are translation-invariant and such that

$$\sum_{j \in \mathbb{Z}^d} J_{ij} < +\infty \tag{3}$$

The free spin measure $\nu(\boldsymbol{\sigma}_i)$ (rotation-invariant) has to be chosen such that

$$\int_{\mathbb{R}^n} \exp(b\boldsymbol{\sigma}_i^2)\, d\nu(\boldsymbol{\sigma}_i) < +\infty \tag{4}$$

for any real b ($n = 1$, 2, or 3). Moreover we shall also require the validity of the Lee–Yang theorem; that is, for $n = 1$ or 2,[13-16] the function

$$g(z) = \int_{\mathbb{R}^n} \exp(z\sigma_i^1)\, d\nu(\boldsymbol{\sigma}_i) \tag{5}$$

has only pure imaginary zeros, and for $n = 3$, we explicitly choose[23]

$$d\nu(\boldsymbol{\sigma}) = \delta(\boldsymbol{\sigma}^2 - 1)\, d\boldsymbol{\sigma} \tag{6}$$

or

$$d\nu(\boldsymbol{\sigma}) = \exp(-\lambda\, |\boldsymbol{\sigma}|^4 + \mu\, |\boldsymbol{\sigma}|^2)\, d\boldsymbol{\sigma} (\lambda > 0,\, \mu \in \mathbb{R}) \tag{7}$$

Let us now introduce the magnetization variable M_Λ associated to the block $\Lambda \subset \Lambda'$:

$$M_\Lambda = \sum_{i \in \Lambda} \sigma_i^1 \tag{8}$$

Since the joint probability distribution of the configurations $(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2,..., \boldsymbol{\sigma}_{|\Lambda'|})$ is given by the finite-volume Gibbs state, (8) clearly represents a sum of dependent random variables. We shall, however, see that above the critical temperature, the dependence is weak enough to ensure the validity of the classical central limit theorem:

$$M_\Lambda/|\Lambda|^{1/2} \xrightarrow[\Lambda, \Lambda' \uparrow \mathbb{Z}^d]{\text{weakly}} \text{Gaussian}$$

As stated in the introduction, this is known to occur when the external field is nonzero.[4]

In the following we shall examine the $h = 0$ case.

3. THE RESULTS

Let us first notice that the existence of the pressure $p(\beta, h)$ (thermodynamic limit) has been established for the class of models we consider here,[10,11] where $p(\beta, h)$ is defined, for instance, by

$$p(\beta, h) = \lim_{\Lambda' \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda'|} \log Z_{\Lambda',\Lambda'}(\beta, h) \tag{9}$$

The limit $\Lambda' \uparrow \mathbb{Z}^d$ has to be taken in the van Hove sense as usual.

For $n=1$ and $n=2$, it is also known[10–12] that there exists one translation-invariant, infinite-volume Gibbs state as $\Lambda' \uparrow \mathbb{Z}^d$ whenever the spontaneous magnetization $(\partial p/\partial h)(\beta, 0)$ exists. That is, for any sequence $\Lambda'_m \uparrow \mathbb{Z}^d$

$$\lim_{\Lambda'_m \uparrow \mathbb{Z}^d} \langle \cdot \rangle^{\beta,0}_{\Lambda'_m} = \langle \cdot \rangle^{\beta,0}_{\mathbb{Z}^d} \tag{10}$$

where

$$\langle \cdot \rangle^{\beta,0}_{\Lambda'_m}$$

denotes the mean value of $\cdot$ with respect to (1) at temperature β^{-1} and zero external field. For $n=3$, one may use the classical argument of compactness to show that there exists, at least for a subsequence (Λ'_{m_k}), an infinite-volume Gibbs state such that the following limit exists:

$$\lim_{\Lambda'_{m_k} \uparrow \mathbb{Z}^d} \langle \cdot \rangle^{\beta,0}_{\Lambda'_{m_k}} = \langle \cdot \rangle^{\beta,0}_{\mathbb{Z}^d} \tag{11}$$

In order to show that

$$M_\Lambda/|\Lambda|^{1/2} \xrightarrow[\Lambda,\Lambda' \uparrow \mathbb{Z}^d]{} \text{Gaussian} \tag{12}$$

in the sense of weak convergence of probability distributions, it remains to study the following limit for any real t:

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \langle \exp(tM_\Lambda/|\Lambda|^{1/2}) \rangle^{\beta,0}_{\mathbb{Z}^d} \tag{13}$$

This is precisely the aim of the following result.

Theorem 1. For the ferromagnets defined in Eqs. (1)–(7), if the temperature is such that

$$\sup_{\Lambda} \frac{1}{|\Lambda|} \langle M_\Lambda^2 \rangle^{\beta,0}_{\mathbb{Z}^d} < +\infty \tag{14}$$

and

$$\lim_{\Lambda\uparrow\mathbb{Z}^d}\frac{1}{|\Lambda|}\langle M_\Lambda^2\rangle_{\mathbb{Z}^d}^{\beta,0}=\frac{\partial^2 p}{\partial h^2}(\beta,0) \tag{15}$$

then, for any real t,

$$\lim_{\Lambda\uparrow\mathbb{Z}^d}\langle \exp(tM_\Lambda/|\Lambda|^{1/2})\rangle_{\mathbb{Z}^d}^{\beta,0}=\exp(t^2\chi/2) \tag{16}$$

where

$$\chi=\partial_h^2 p(\beta,0) \tag{17}$$

Remarks. 1. If $n=1$, hypothesis (14) reduces to Newman's hypothesis concerning the finiteness of the susceptibility as stated in Refs. 5 and 6. The validity of (15) can be established by using, for instance, FKG inequalities.[24]

2. For $n=3$, a subsequence Λ_{m_j} has to be considered. Since the corresponding modifications in our formulas are obvious, we do not burden the notation.

Let us first give some technical results, the proofs of which can be found in the Appendix.

Lemma 1. For the ferromagnets defined in Eqs. (1)–(7), there exists an infinitely divisible characteristic function $\Phi_\Lambda(t)$ such that for any real t and for any fixed β

$$\langle \exp(tM_\Lambda)\rangle_{\mathbb{Z}^d}^{\beta,0}=1/\Phi_\Lambda(t) \tag{18}$$

Lemma 2. With the previously defined notations and for the class of ferromagnets we consider [Eqs. (1)–(7)], if

$$\sup_\Lambda\frac{1}{|\Lambda|}\langle M_\Lambda^2\rangle_{\mathbb{Z}^d}^{\beta,0}<+\infty \tag{19}$$

and

$$\lim_{\Lambda\uparrow\mathbb{Z}^d}\frac{1}{|\Lambda|}\langle M_\Lambda^2\rangle_{\mathbb{Z}^d}^{\beta,0}=\frac{\partial^2 p}{\partial h^2}(\beta,0) \tag{20}$$

then for any real t with $\tilde{t}=t/|\Lambda|^{1/2}$

$$\lim_{\Lambda\uparrow\mathbb{Z}^d}(\langle M_\Lambda;M_\Lambda\rangle_{\mathbb{Z}^d}^{\beta,\tilde{t}}/\langle M_\Lambda;M_\Lambda\rangle_{\mathbb{Z}^d}^{\beta,0})=1 \tag{21}$$

where

$$\langle M_\Lambda; M_\Lambda \rangle_{\mathbb{Z}^d}^{\beta,h} = \langle M_\Lambda^2 \rangle_{\mathbb{Z}^d}^{\beta,h} - (\langle M_\Lambda \rangle_{\mathbb{Z}^d}^{\beta,h})^2 \tag{22}$$

Proof of Theorem 1. According to Lemma 1, one knows that there exists an infinitely divisible characteristic function ϕ_Λ such that for any real t

$$\langle \exp(tM_\Lambda/|\Lambda|^{1/2}) \rangle_{\mathbb{Z}^d}^{\beta,0} = 1/\phi_\Lambda(t/|\Lambda|^{1/2}) \tag{23}$$

Let us now assume that the variance of the corresponding probability distribution is finite for any $\Lambda \subset \mathbb{Z}^d$. This means

$$-\phi_\Lambda''(0) = \frac{1}{|\Lambda|} \langle M_\Lambda^2 \rangle_{\mathbb{Z}^d}^{\beta,0} < +\infty \tag{24}$$

where the double prime indicates the second derivative with respect to t. This implies a restriction on the temperature range. More explicitly, we shall assume the validity of (14); then (24) will clearly be satisfied. It should be stressed that if the variance of the magnetization variable exists as $\Lambda'_{m_k} \uparrow \mathbb{Z}^d$, the mean value of the magnetization has to be zero at vanishing external field. This allows us to use the results of Refs. 10–12. This corresponds in fact to temperatures β^{-1} above the critical temperature β_c^{-1} defined by

$$\beta_c = \sup\{\beta \in \mathbb{R}_0^+ : \sup_\Lambda \frac{1}{|\Lambda|} \langle M_\Lambda^2 \rangle_{\mathbb{Z}^d}^{\beta,0} < +\infty\} \tag{25}$$

With this restriction on temperature, one can easily prove[7] that there exists a bounded, nondecreasing function $K_\Lambda(x)$ with $K_\Lambda(-\infty) = 0$ (Kolmogorov's spectral function[1]) such that

$$\log\langle \exp(\tilde{t} M_\Lambda) \rangle_{\mathbb{Z}^d}^{\beta,0} = \int_{\mathbb{R}} \frac{1 - \cos(\tilde{t}x)}{x^2} \, dK_\Lambda(x) \tag{26}$$

the second derivative with respect to t of which leads to

$$\langle M_\Lambda; M_\Lambda \rangle_{\mathbb{Z}^d}^{\beta,\tilde{t}} = \int \cos(\tilde{t}x) \, dK_\Lambda(x) \tag{27}$$

One therefore has that for any t

$$\langle M_\Lambda; M_\Lambda \rangle_{\mathbb{Z}^d}^{\beta,t} \leqslant \langle M_\Lambda; M_\Lambda \rangle_{\mathbb{Z}^d}^{\beta,0} \tag{28}$$

Hypothesis (14) thus ensures that the logarithm of the lhs of (23) admits a Taylor approximation of order 2:

$$\log\langle \exp(tM_\Lambda/|\Lambda|^{1/2})\rangle_{\mathbb{Z}^d}^{\beta,0} = \tfrac{1}{2}t^2\langle M_\Lambda; M_\Lambda\rangle_{\mathbb{Z}^d}^{\beta,\theta_\Lambda \bar{t}}/|\Lambda| \tag{29}$$

with $0 < \theta_\Lambda < 1$. To establish the validity of (16), one has to prove that for any real t

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \langle M_\Lambda; M_\Lambda\rangle_{\mathbb{Z}^d}^{\beta,\bar{t}} = \partial_h^2 p(\beta, 0) \tag{30}$$

Assuming the validity of (15), it remains to show that for any real t

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} (\langle M_\Lambda; M_\Lambda\rangle_{\mathbb{Z}^d}^{\beta,\bar{t}}/\langle M_\Lambda; M_\Lambda\rangle_{\mathbb{Z}^d}^{\beta,0}) = 1 \tag{31}$$

This result has precisely been established in Lemma 2. This achieves the proof. ∎

4. GENERALITY OF THE RESULTS

Let us finally comment on the generality of the result reported in this paper. The Gaussian nature of the magnetization fluctuations for Lee–Yang ferromagnets reduces now to the following analysis:

1. If $h \neq 0$, then one can use Iagolnitzer and Souillard's result, as already stated in the introduction.
2. If $h = 0$, then one has to prove the validity of our hypotheses (14) and (15).

It seems to us that hypothesis (14) is quite natural, since it only requires the finiteness of the susceptibility. This simply means that we consider a temperature above the critical temperature as defined in (25). Hypothesis (15) is relative to a fluctuation-dissipation relation, but it only requires the validity of this property for the integrated two-point function. It has, however, not yet been possible to establish the validity of (15) using the Lee–Yang theorem only.

Let us point out that Lebowitz has been able to establish this fluctuation-dissipation relation using FKG inequalities above the critical temperature[25] and that De Coninck and Dunlop[9] extended this result to multicomponent ferromagnets using Simon[18,19] and Gaussian[20–22] inequalities. This already ensures the validity of the Gaussian fluctuation theorem for classical Heisenberg ferromagnets above the critical temperature at zero external field.

Finally, we would like to point out that our method of proof can also be easily applied when the external field is nonzero. We do not report this calculation here, since the result is already known.[4]

APPENDIX

Proof of Lemma 1. By definition, one has, with $\tilde{t} = t/|\Lambda|^{1/2}$,

$$\langle \exp(\tilde{t} M_\Lambda) \rangle_{\Lambda'}^{\beta,0} = \int_{\mathbb{R}^{|\Lambda'|}} \exp(\tilde{t} M_\Lambda)\, d\mu_{\Lambda,\Lambda'}((\boldsymbol{\sigma}_j)_{j \in \Lambda'} \mid \beta, 0) \tag{A.1}$$

which, using (2), leads to

$$\langle \exp(\tilde{t} M_\Lambda) \rangle_{\Lambda'}^{\beta,0} = Z_{\Lambda,\Lambda'}(\beta, \tilde{t}) / Z_{\Lambda,\Lambda'}(\beta, 0) \tag{A.2}$$

The Lee–Yang theorem[13–17] now guarantees that, as a function of $\tilde{t}$, the rhs of (A.2) has only pure imaginary zeros. This allows us to prove[7] that there exists an infinitely divisible characteristic function $\phi_{\Lambda,\Lambda'}$ (Fourier transform of a probability measure) such that for any real $\tilde{t}$

$$\frac{Z_{\Lambda,\Lambda'}(\beta, \tilde{t})}{Z_{\Lambda,\Lambda'}(\beta, 0)} = \frac{1}{\phi_{\Lambda,\Lambda'}(\tilde{t})} \tag{A.3}$$

Let us now consider the limit $\Lambda' \uparrow \mathbb{Z}^d$. Using the classical argument of compactness, there exists, at least for a subsequence (Λ'_{m_k}), an infinite-volume Gibbs state

$$\mu_{\Lambda, \mathbb{Z}^d}((\boldsymbol{\sigma}_j)_{j \in \mathbb{Z}^d} \mid \beta, 0)$$

such that

$$\lim_{\Lambda'_{m_k} \uparrow \mathbb{Z}^d} \langle \exp(\tilde{t} M_\Lambda) \rangle_{\Lambda'_{m_k}}$$

$$= \int_{\mathbb{R}^{\mathbb{Z}^d}} \exp\left(\tilde{t} \sum_{i \in \Lambda} \sigma_i^1\right) d\mu_{\Lambda, \mathbb{Z}^d}((\boldsymbol{\sigma}_j)_{j \in \mathbb{Z}^d} \mid \beta, 0) \tag{A.4}$$

By convexity, it is easily seen that the rhs of this expression is a continuous function of t. This ensures that the limit

$$\lim_{\Lambda'_{m_k} \uparrow \mathbb{Z}^d} \phi_{\Lambda,\Lambda'_{m_k}}(t) \tag{A.5}$$

exists and is a continuous function of t. By the continuity theorem,[2] one therefore knows that there exists a characteristic function $\phi_\Lambda(t)$ such that

$$\lim_{\Lambda'_{m_k} \uparrow \mathbb{Z}^d} \phi_{\Lambda,\Lambda'_{m_k}}(t) = \phi_\Lambda(t) \tag{A.6}$$

for any real t. That this function $\phi_{\Lambda}(t)$ is also infinitely divisible follows easily from the property (Ref. 2, p. 110) that "a characteristic function which is the limit of a sequence of infinitely divisible characteristic functions is infinitely divisible." ∎

Proof of Lemma 2. We shall in fact prove that the ratio

$$f_{\Lambda}(t) = \langle M_{\Lambda}; M_{\Lambda} \rangle_{\mathbb{Z}^d}^{\beta,t} / \langle M_{\Lambda}; M_{\Lambda} \rangle_{\mathbb{Z}^d}^{\beta,0} \tag{A.7}$$

is a characteristic function and the limit as $\Lambda \uparrow \mathbb{Z}^d$ is also a characteristic function. Knowing that the convergence of a sequence of characteristic functions is uniform in every finite interval,[2] one easily gets (21), since $\tilde{t} = t/|\Lambda|^{1/2}$ is going to zero as $\Lambda \uparrow \mathbb{Z}^d$.

Let us prove that the ratio $f_{\Lambda}(t)$ is a characteristic function.

Using the classical representation formula of the logarithm of an infinitely divisible characteristic function,[1,2,7,8] one gets, taking the second derivative with respect to t of the logarithm of (18),

$$\langle M_{\Lambda}; M_{\Lambda} \rangle_{\mathbb{Z}^d}^{\beta,t} = \int_{\mathbb{R}} \cos(tx)\, dK_{\Lambda}(x) \tag{A.8}$$

from which one deduces that

$$\langle M_{\Lambda}; M_{\Lambda} \rangle_{\mathbb{Z}^d}^{\beta,0} = K_{\Lambda}(+\infty) \tag{A.9}$$

This leads to the identity

$$f_{\Lambda}(t) = \int_{\mathbb{R}} \cos(tx)\, d\frac{K_{\Lambda}(x)}{\int dK_{\Lambda}(x)} \tag{A.10}$$

Since $K_{\Lambda}(x)$ is a bounded, nondecreasing function, this function $f_{\Lambda}(t)$ can indeed be identified with a characteristic function. This achieves the first step of the proof.

Let us now consider what happens as $\Lambda \uparrow \mathbb{Z}^d$. One gets

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} f_{\Lambda}(t) = \left(\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \langle M_{\Lambda}; M_{\Lambda} \rangle_{\mathbb{Z}^d}^{\beta,t} \right) \Big/ \left(\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \langle M_{\Lambda}^2 \rangle_{\mathbb{Z}^d}^{\beta,0} \right) \tag{A.11}$$

Using hypotheses (19) and (20) and the Lee–Yang theorem, one finds that

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} (\langle M_{\Lambda}; M_{\Lambda} \rangle_{\mathbb{Z}^d}^{\beta,t} / \langle M_{\Lambda}^2 \rangle_{\mathbb{Z}^d}^{\beta,0}) = \partial_h^2 p(\beta, t) / \partial_h^2 p(\beta, 0) \tag{A.12}$$

That this limit is itself a characteristic function requires a proof of its continuity at $t = 0$. This can easily be done using the result of Ref. 8. Indeed, one knows that for the class of models we consider the thermodynamic limit of the pressure exists and is a continuous function of the external field:

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} [\langle \exp(tM_\Lambda) \rangle_\Lambda^{\beta,0}]^{1/|\Lambda|} = \exp\{p(\beta, t) - p(\beta, 0)\} \tag{A.13}$$

This implies that there exists an infinitely divisible characteristic function $\Psi(t)$ such that[8]

$$\exp\{p(\beta, t) - p(\beta, 0)\} = 1/\Psi(t) \tag{A.14}$$

One deduces from this formula that

$$p(\beta, t) = p(\beta, 0) - \log \Psi(t) \tag{A.15}$$

and therefore

$$\partial_t^2 p(\beta, t) = -\frac{\Psi''(t)}{\Psi(t)} + \left[\frac{\Psi'(t)}{\Psi(t)}\right]^2 \tag{A.16}$$

This means that there exists a probability distribution function $H(x)$ such that

$$\begin{aligned} \partial_t^2 p(\beta, t) = & \left\{ \int x^2 \cos(tx)\, dH(x) \int \cos(tx)\, dH(x) \right. \\ & \left. - \left[\int x \sin(tx)\, dH(x) \right]^2 \right\} \\ & \times \left[\int \cos(tx)\, dH(x) \right]^{-2} \end{aligned} \tag{A.17}$$

That $\partial_t^2 p(\beta, t)$ is a continuous function of t follows easily from the existence of $\partial_t^2 p(\beta, 0)$.

One has therefore proven that $f_\Lambda(t)$ is a sequence of characteristic functions that converges as $\Lambda \uparrow \mathbb{Z}^d$ to a characteristic function. Since the convergence of a sequence of characteristic function is uniform with respect to the variable t in every finite interval,[2] we easily achieve the proof of the lemma by noticing that

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} f_\Lambda(t/|\Lambda|^{1/2}) = \lim_{\Lambda \uparrow \mathbb{Z}^d} f_\Lambda(0) = 1 \quad \blacksquare \tag{A.18}$$

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